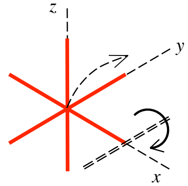
Problem 10.33



Three rods of length "L" each are welded to one another at their centers and at right angles as shown in the sketch. Someone wants to rotate the system about an axis that is at the end of one bar, parallel to the *y-axis* and perpendicular to the *x-axis* (this is shown with a double dashed line in the sketch).

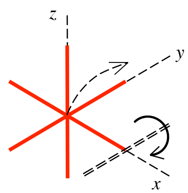
a.) What is the *moment of inertia* about the axis of rotation?

This is one of those cases when you could fake yourself out if not careful. Specifically, you could try to set up

$$\int r^2\,dm$$

for each of the rods, relative to the rotational axis (this would, I might add, but a real pain in the arse, but I'll do it for the nerds in the crowd at the end of the problem). An alternate way to go would be to determine the *moment of inertia* about the system's *center of mass* and along the *y-axis*, then use the *Parallel Axis Theorem* to determine the *moment of inertia* about the axis of rotation *parallel* to the *center of mass's I*. That is the way we will go.

About the *center of mass*:



1.) Because "r" is zero, the *moment of inertia* about *y-axis* do to the rod oriented *along* the *y-axis* is ZERO.

2.) Looking at the handy-dandy table in the text, the *moment of inertia* of a rod that is perpendicular to and centered on the *y-axis* (this is the case for both bars) is:

$$I_{cm} = \frac{1}{12} mL^2$$

The total *moment of inertia* about the *y-axis* is therefore:

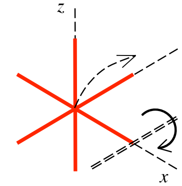
$$I_{total} = \frac{1}{12} mL^2 + 0 + \frac{1}{12} mL^2$$
$$= \frac{1}{6} mL^2$$

The *Parallel Axis Theorem* states that:

$$I_{rotAxis} = I_{cm} + Md^2$$

where "M" is the total mass in the system and "d" is the distance between the *center of mass axis* and the *parallel axis*.

Using this:



$$I_{rotAxis} = I_{cm} + Md^2$$
$$= \left(\frac{1}{6} mL^2\right) + \left[(3m)\left(\frac{L}{2}\right)^2\right]$$
$$= \frac{11}{12} mL^2$$

FIN: This concludes the problem unless you want some more information about the conceptual side of *moment of inertia* quantities, and want to see the math associate with doing this problem the hard way.

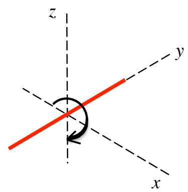
EXTRA (and if the verbiage is too much, go directly to *Page 6*)

You have gotten to the place where you can do *moment of inertia* calculations, but do you really understand what the *moment of inertia* does for you in a conceptual sense (understanding this actually has value—I've seen AP questions that test your conceptual grasp of this, versus your mathematical competence).

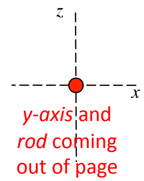
Consider: You are out in space sitting next to a 1966 Volkswagen and a box of Kleenex (Handy Wipes, whatever). Between the two, why is the Volkswagen said to have more mass? Because the Volkswagen has more *resistance to changing its motion,* more *inertia*. That is what mass measures; it's a relative measure of a body's *resistance to changing its motion*, or *inertia*!

Objects that rotate about some axis also have *resistance to changing their rotational motion* about that axis. Ignoring the fact that some bright soul decided to call the quantity that measures that rotational inertia *moment of inertia* (versus the obvious in *rotational inertia*), what is different about this quantity is that it is not just governed by how much *mass* there is in the object, but also *how the mass is distributed about the rotational axis.*

So, for instance, let's say you have a rod of mass M and length L sitting along the *y-axis*, and you want to spin it about an axis through its *center of mass* and perpendicular to the *z-axis* in the *x-y plane*. The bits of mass that are in close to the *z-axis* will contribute very little to the resistance the rod experiences when it tries to change its rotational motion about that axis, whereas the bits of mass that are far out will contribute a lot. So, as you have recently learned, the calculated, net *moment of inertia* for this situation is $I_{rod} = \frac{1}{12}mL^2$.

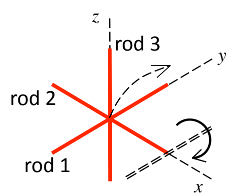


But what would happen if you decided you wanted to rotate the rod about an axis through its center of mass and along the *y-axis*? Now all of the mass is very close to the axis of rotation, essentially along it, and the *moment of inertia* is tiny.



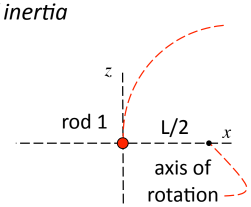
This is why the original conceptual description of how to determine a *moment of inertia* said, "Move out from the rotational axis until you find some mass. Take that mass and multiply it by the square of the distance out. Do that for all the mass bits in the system and sum them, and you will end up with the body's *moment of inertia* about that axis!"

So with all that in mind, what's going on with our system in the raw? (Note that I've numbered the rods for easy commentary.)



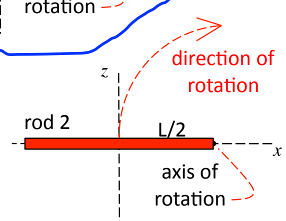
Rod 1: This rod is parallel to the axis of rotation with all the mass "L/2" units away from that axis (see sketch). As such, that *moment of inertia* quantity is:

$$I_{rod1} = m\left(\frac{L}{2}\right)^2 = \frac{1}{4}mL^2$$

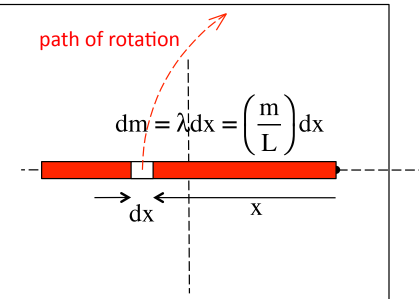


Easy peasie . . .

Rod 2: Not so easy! Looking down the *y-axis*, the system looks like the sketch to the right. So now we need to take a differential bit of mass "dm" comprising a differentially long section of the rod "dx" located an arbitrary distance "x" units from the axis of rotation, and . . .



OK, look at the more detailed sketch of the situation for Rod 2. With it, we can write:

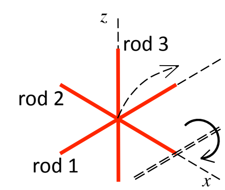


$$\begin{aligned} I_{rod2} &= \int r^2\,dm \\ &= \int_{x=0}^{L} x^2(\lambda dx) \\ &= \int_{x=0}^{L} x^2\left(\frac{m}{L}dx\right) \\ &= \left(\frac{m}{L}\right)\left(\frac{x^3}{3}\right)\Big|_{x=0}^{L} = \left(\frac{m}{L}\right)\left(\frac{L^3}{3}\right) \\ &= \frac{1}{3}mL^2 \end{aligned}$$

Again, not too horrendous if you get it pictured appropriately and remember what you are trying to do. Also, if you are familiar with the table in the text, this is the correct *moment of inertia* for a rod about its end!

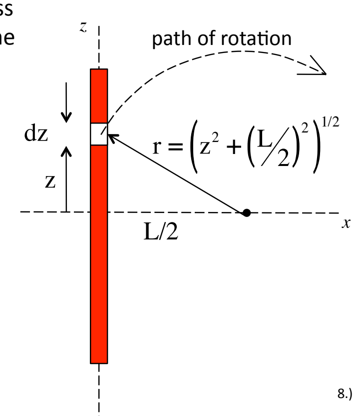
Rod 3 (and the kicker): Now it gets sticky. The sketch shows the situation. It is still true that the differential mass in any given differential section will equal:

$$dm = \lambda dz = \left(\frac{m}{L}\right)dz$$

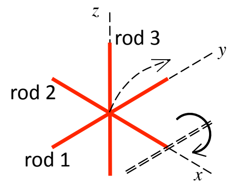


What is nasty is that each bit of differential mass "dm" has a different radius of motion. Doing the upper half and doubling, we can write:

$$\begin{aligned} I_{rod3} &= \int r^2\,dm \\ &= 2\int_{z=0}^{L/2}\left[\left(z^2 + \left(\tfrac{L}{2}\right)^2\right)^{1/2}\right]^2(\lambda dz) \\ &= 2\int_{z=0}^{L/2}\left(z^2 + \left(\tfrac{L}{2}\right)^2\right)\left(\left(\frac{m}{L}\right)dz\right) \\ &= 2\left(\frac{m}{L}\right)\left[\int_{z=0}^{L/2} z^2\,dz + \left(\frac{L}{2}\right)^2\int_{z=0}^{L/2} dz\right] \end{aligned}$$



Evaluating this, we get:



$$I_{rod\,3} = 2\left(\frac{m}{L}\right)\left[\int_{z=0}^{L/2} z^2\,dz + \left(\frac{L}{2}\right)^2 \int_{z=0}^{L/2} dz\right]$$

$$= 2\left(\frac{m}{L}\right)\left[\left(\frac{z^3}{3}\right)\Big|_{z=0}^{L/2} + \left(\frac{L^2}{4}\right)z\Big|_{z=0}^{L/2}\right]$$

$$= 2\left(\frac{m}{L}\right)\left[\left(\frac{L^3}{24}\right) + \left(\frac{L^3}{8}\right)\right]$$

$$= \frac{1}{3}mL^2$$

So . . .

$$I_{total} = I_{rod1} + I_{rod2} + I_{rod\,3}$$

$$= \frac{1}{4}mL^2 + \frac{1}{3}mL^2 + \frac{1}{3}mL^2$$

$$= \frac{11}{12}mL^2 \qquad \text{(Q.E.D.)}$$